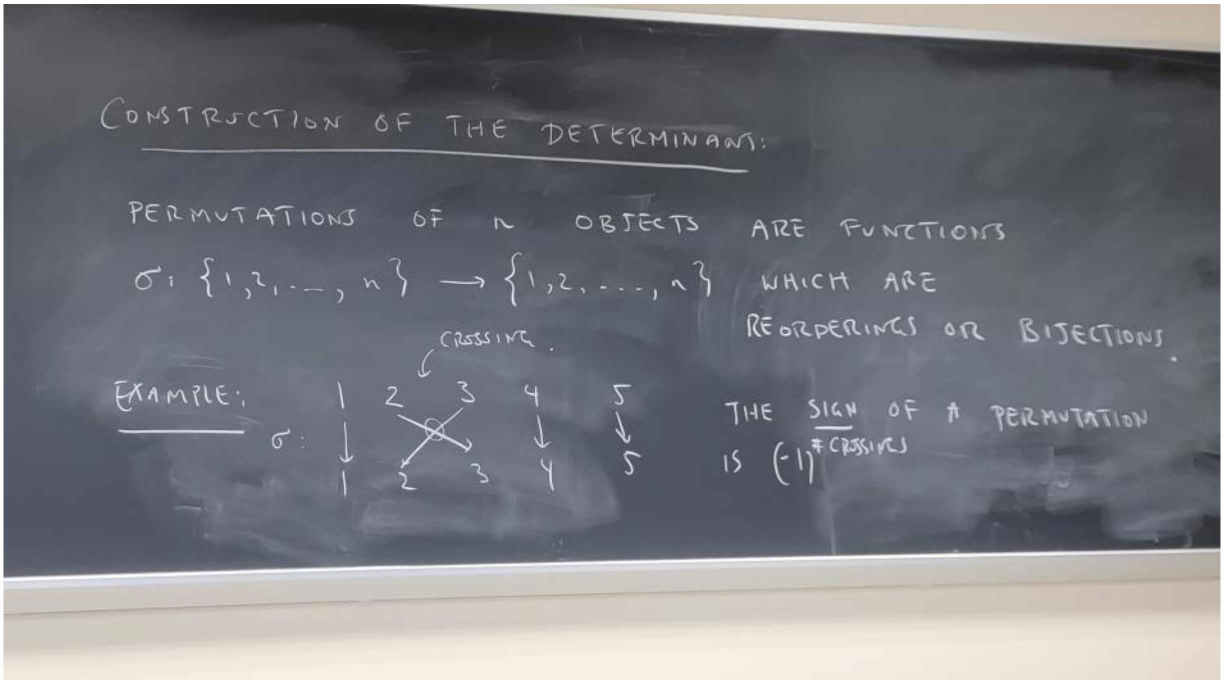
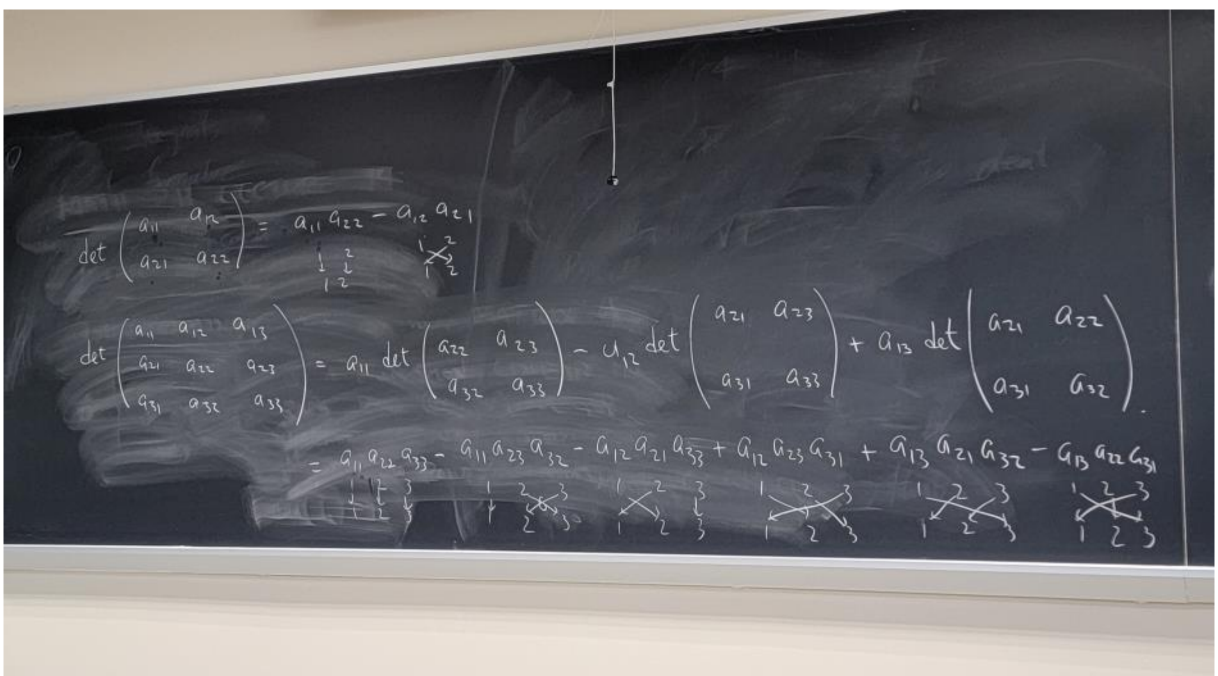
3/21/23

Monday, March 27, 2023 7:00 PM





DEFINITION: THE DETERMINANT OF AN $n \times n$ MATRIX A IS THE SUM

$$\sum_{\substack{\sigma \text{ PERMUTATION} \\ \text{OF } \{1,2,\dots,n\}}} \operatorname{sign}(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

THEOREM: THIS DEFINITION OF THE DETERMINANT SATISFIES THE REQUIRED PROPERTIES.

PROOF: $\det\begin{pmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ & & & 1 & 0 \\ 0 & \cdots & & 0 & 1 \end{pmatrix}$

$\sigma(1)=1, \sigma(2)=2, \dots, \sigma(n)=n.$

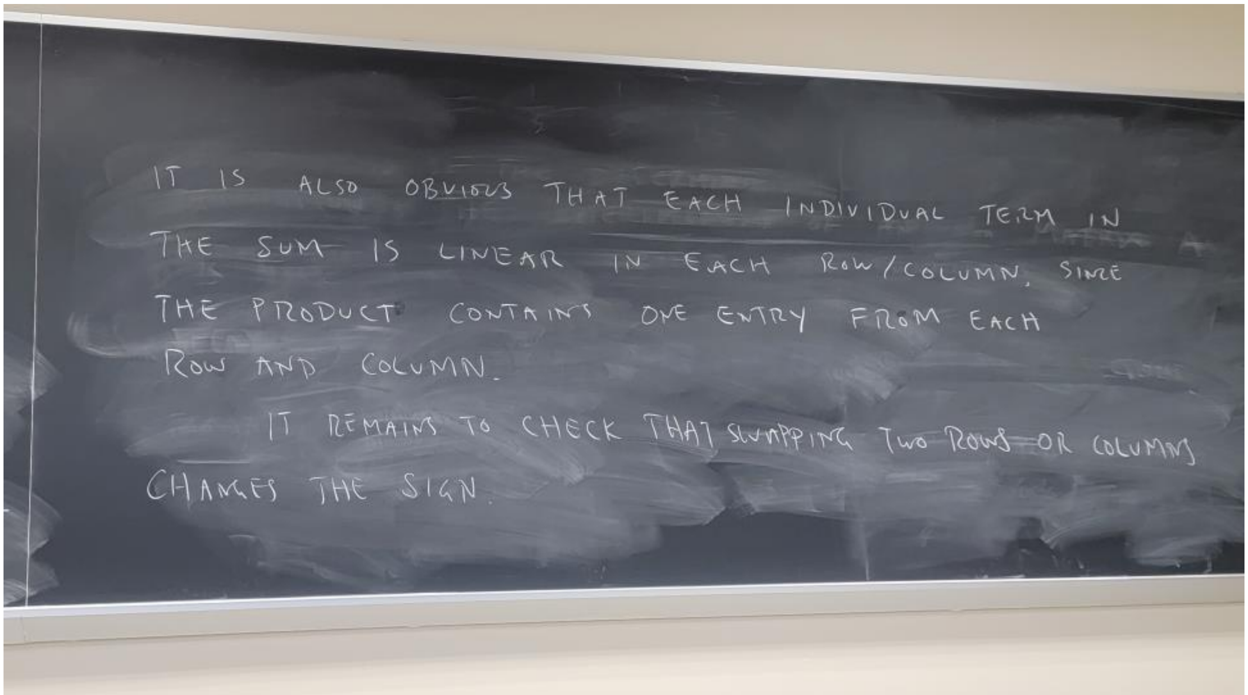
NO CROSSINGS $\Rightarrow \det(I_n) = 1.$

IN THE SUM THAT IS FORMED, EACH TERM SELECTS EXACTLY ONE ENTRY FROM EACH ROW AND COLUMN

$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ IN THE IDENTITY, EACH DIAGONAL ENTRY MUST BE SELECTED, OR THE PRODUCT IS 0.

IT IS ALSO OBVIOUS THAT EACH INDIVIDUAL TERM IN THE SUM IS LINEAR IN EACH ROW/COLUMN, SINCE THE PRODUCT CONTAINS ONE ENTRY FROM EACH ROW AND COLUMN.

IT REMAINS TO CHECK THAT SWAPPING TWO ROWS OR COLUMNS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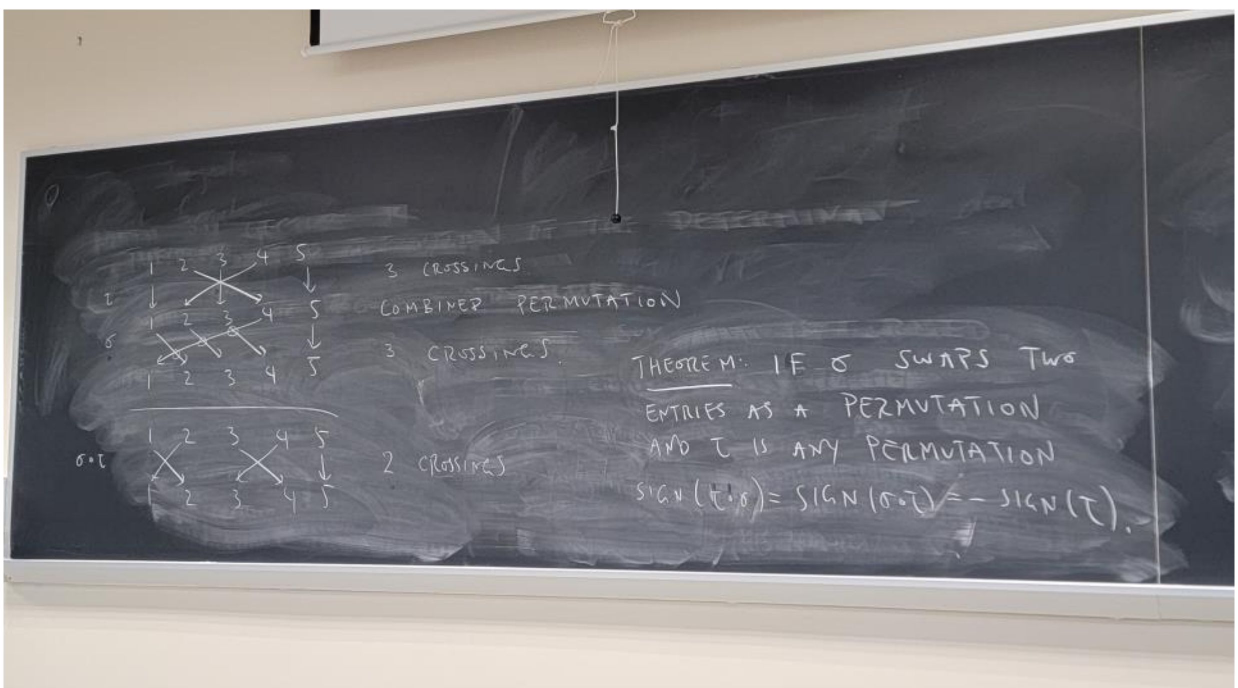
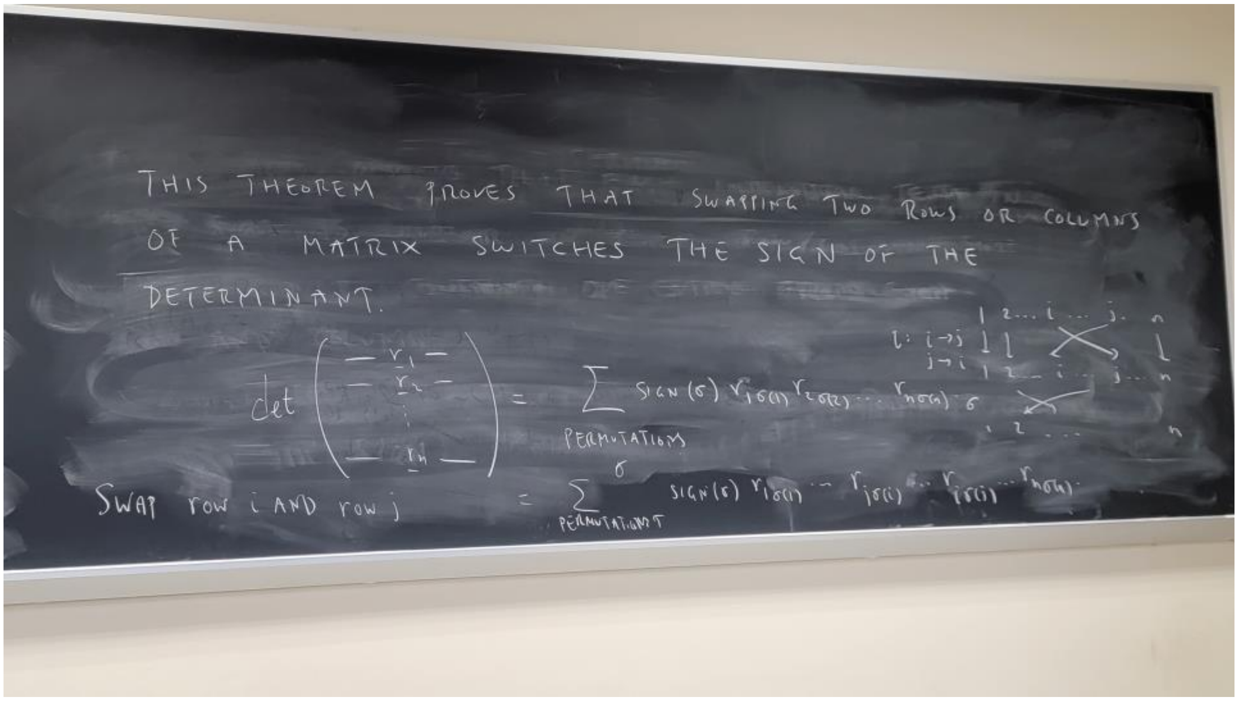
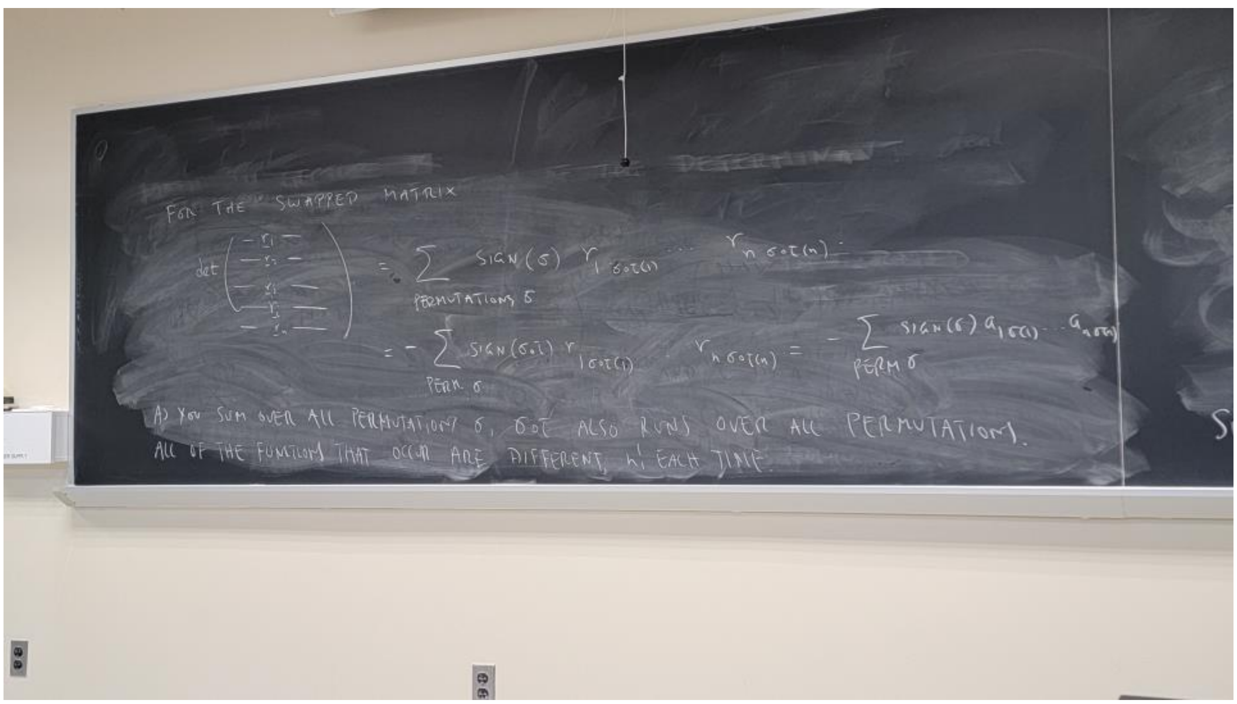
THE PRODUCT CONTAINS ONE ENTRY FROM EACH
ROW AND COLUMN.
IT REMAINS TO CHECK THAT SWAPPING TWO ROWS OR COLUMNS
CHANGES THE SIGN.


1 2 3 4 5
τ
1 2 3 4 5
σ
1 2 3 4 5
3 CROSSINGS
COMBINED PERMUTATION
3 CROSSINGS.
σ∘τ
1 2 3 4 5
1 2 3 4 5
2 CROSSINGS
THEOREM: IF σ SWAPS TWO
ENTRIES AS A PERMUTATION
AND τ IS ANY PERMUTATION
SIGN(τ∘σ) = SIGN(σ∘τ) = − SIGN(τ).


THIS THEOREM PROVES THAT SWAPPING TWO ROWS OR COLUMNS OF A MATRIX SWITCHES THE SIGN OF THE DETERMINANT.
$\det\begin{pmatrix} — r_1 — \\ — r_2 — \\ \vdots \\ — r_n — \end{pmatrix} = \sum_{\text{PERMUTATIONS } \sigma} \text{SIGN}(\sigma)\, r_{1\sigma(1)} r_{2\sigma(2)} \cdots r_{n\sigma(n)}$
SWAP ROW i AND ROW j
$= \sum_{\text{PERMUTATIONS } \tau} \text{SIGN}(\sigma)\, r_{1\sigma(1)} \cdots r_{j\sigma(i)} \cdots r_{i\sigma(j)} \cdots r_{n\sigma(n)}$
τ: $i \to j$, $j \to i$


FOR THE SWAPPED MATRIX
$\det\begin{pmatrix} — r_1 — \\ — r_2 — \\ — r_j — \\ — r_i — \\ — r_n — \end{pmatrix} = \sum_{\text{PERMUTATIONS } \sigma} \text{SIGN}(\sigma)\, r_{1\,\sigma\circ\tau(1)} \cdots r_{n\,\sigma\circ\tau(n)}$
$= -\sum_{\text{PERM } \sigma} \text{SIGN}(\sigma\circ\tau)\, r_{1\,\sigma\circ\tau(1)} \cdots r_{n\,\sigma\circ\tau(n)} = -\sum_{\text{PERM } \sigma} \text{SIGN}(\sigma)\, a_{1\sigma(1)} \cdots a_{n\sigma(n)}$
AS YOU SUM OVER ALL PERMUTATIONS σ, $\sigma\circ\tau$ ALSO RUNS OVER ALL PERMUTATIONS.
ALL OF THE FUNCTIONS THAT OCCUR ARE DIFFERENT, $n!$ EACH TIME.

A SIMILAR ARGUMENT SHOWS THAT SWAPPING TWO COLUMNS CHANGES THE SIGN OF THE DETERMINANT.

$$\det \begin{array}{c} \text{col } i \quad \text{col } j \\ \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \end{array}$$

IF COLUMNS i, j ARE SWAPPED, THE RESULTING SUM IS $\displaystyle\sum_{\text{PERM } \sigma} \text{SIGN}(\sigma)\, a_{1(\tau\circ\sigma)(1)} \cdots a_{n(\tau\circ\sigma)(n)}$

τ SWAPS i, j.

THIS ALSO FLIPS THE SIGN.

THE PROOF OF THE FACT REGARDING SIGNS GOES AS FOLLOWS.

$i < j$

TOTAL # CROSSINGS: CROSSINGS NOT INVOLVING i AND j (UNCHANGED BY SWAPPING i, j)

+ CROSSINGS $t < i$, $\sigma(t) > \sigma(i)$, $i < t < j$, $\sigma(t) < \sigma(i)$, $t > j$, $\sigma(t) < \sigma(i)$. (ARROW FROM i CROSSED)

THEOREM: $\det(A) = \det(A^T)$.

PROOF: THE FORMULA WE GAVE IS LINEAR IN ROWS AND COLUMNS

ALTERNATES SWAPPING ROWS OR COLUMNS SO CAN BE VIEWED AS THE SAME FUNCTION OF THE ROWS OR COLUMNS.

THEOREM: $\det(AB) = \det(A)\det(B)$.

PROOF: WE CHECKED THIS WHEN MULTIPLYING ON THE LEFT OR RIGHT BY ELEMENTARY MATRICES, WHICH CORRESPOND TO:

1. MULTIPLYING A ROW OR COLUMN BY A CONSTANT
2. SWAPPING A ROW OR COLUMN
3. ADDING A MULTIPLE OF ONE ROW OR COLUMN TO ANOTHER

ANY INVERTIBLE MATRIX CAN BE WRITTEN AS THE PRODUCT OF ELEMENTARY MATRICES SO THE FORMULA HOLDS FOR INVERTIBLE MATRICES.

IF A OR B IS NOT INVERTIBLE THEIR PRODUCT IS NOT AND THE DETERMINANTS ARE 0.

THEOREM: $\det(A^{-1}) = \dfrac{1}{\det A}$

PROOF: $A^{-1}A = I_n$

$$\det(A^{-1})\det(A) = 1.$$

THEOREM: IF $A = SBS^{-1}$, $\det A = \det(B)$.

PROOF: $\det A = \det S \det B \det S^{-1}$

$$= \det S \det B \frac{1}{\det S} = \det B. \quad \square$$

DEFINITION: A_{ij} MATRIX WITH iTH ROW, jTH COLUMN DELETED.

LAPLACE EXPANSION:

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).$$